Eur. Phys. J. D **31**, 101–105 (2004)
DOI: 10.1140/epjd/e2004-00110-7

THE EUROPEAN PHYSICAL JOURNAL D

Vortex formation in a slowly rotating Bose-Einstein condensate confined in a harmonic-plus-Gaussian laser trap

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Received 19 February 2004 / Received in final form 24 June 2004
Published online 24 August 2004 –

Abstract. Motivated by the recent experiment at ENS [V. Bretin, S. Stock, Y. Seurin and, J. Dalibard, Phys. Rev. Lett. **92**, 050403 (2004)], we study a rotating (non-)interacting atomic Bose-Einstein condensate confined in a harmonic-plus-Gaussian laser trap potential. By adjusting the amplitude of the Gaussian laser potential, one can make quadratic-plus-quartic potential, purely quartic potential, and quartic-minus-quadratic potential. We show that an interacting Bose-Einstein condensate confined in a harmonic-plus-Gaussian laser trap breaks the rotational symmetry of the Hamiltonian when rotational frequency is greater than one-half of the lowest energy surface mode frequency. We also show that by increasing the amplitude of the Gaussian laser trap, a vortex appears in a slowly rotating Bose-Einstein condensate. Moreover, one can also create a vortex in a slowly rotating non-interacting Bose-Einstein condensate confined in harmonic-plus-Gaussian laser potential.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 05.30.Jp Boson systems

The rotation of a macroscopic quantum fluid exhibits interesting counter-intuitive phenomena. For example, an atomic interacting(repulsive) Bose-Einstein condensate (BEC) confined in a rotating harmonic trap produces quantized vortices for a sufficiently large rotation [1–3]. The theoretical calculation for Ω_c based on purely thermodynamic arguments is significantly smaller than the observed value of Ω_c which is the order of $\sim 0.7\omega_0$, where ω_0 is the radial trap frequency. The vortex formation of a harmonically trapped atomic BEC is related to the dynamical instability of the lowest energy surface mode excitations whose energy scale is set by the harmonic potential [4]. Above a certain rotation frequency of a harmonically symmetric trapped interacting BEC, the system itself starts deforming from a circular shape to an elliptic shape and hence it breaks the original rotational symmetry of the Hamiltonian. This has been predicted only for a harmonically trapped BEC [5] and detected at the ENS experiment which leads to the vortex nucleation [6]. The vortex nucleation starts when the average angular momentum of each particle is one which has been measured experimentally [2].

The harmonically trapped BEC becomes singular when the rotation frequency is equal or greater than the harmonic trap frequency, because the outward centrifugal force counteracts the inward force from the harmonic trap. One can eliminate the singularity at $\Omega \geq 1$ by considering an additional stiffer radial potential (say, quartic potential for simplicity). There is a growing interest about the effect of an anharmonic potential on the properties of a rotating BEC [7–13]. In spite of the fact that the Ω_c in a harmonically trapped BEC calculated from the thermodynamic arguments does not match with the experimental values, many authors have studied Ω_c in a BEC confined in quadratic-plus-quartic trap based on the thermodynamic arguments [8–11]. In this work, we show that a vortex appears in a BEC confined in a quadratic-plus-Gaussian laser potential due to the spontaneous shape deformation of the system as it is happens in a BEC confined by harmonic trap only and calculate the correct critical rotational frequency to create a single vortex. We also show that by increasing the magnitude of the laser trap, a vortex can nucleate in a slowly rotating Bose gas.

Recently, the quadratic-plus-quartic potential has been achieved experimentally by superimposing a blue detuned laser beam to the magnetic trap holding the atoms [15]. The effective external potential is

$$V_t(r,z) = \frac{1}{2} m\omega_0^2 \left[r^2 + \epsilon(x^2 - y^2) + \frac{\omega_z^2}{\omega_0^2} z^2 \right] + U(r), \quad (1)$$

where $\epsilon = (\omega_x^2 - \omega_y^2)/(\omega_x^2 + \omega_y^2)$ is the deformation parameter due to the stirring potential which rotates the system and creates an anisotropic potential in the xy plane. Also, $2\omega_0^2 = \omega_x^2 + \omega_y^2$ and the potential created by the Gaussian laser beam is

$$U(r) = U_0 e^{-\frac{2r^2}{w^2}}. \quad (2)$$

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At the ENS experimental set-up [15], the laser waist is $w = 25$ μm and the amplitude of the laser beam is $U_0 \sim 1.242 \times 10^{-30}$ J. The harmonic trap frequencies are $\omega_0 = 2\pi \times 75.5$ Hz and $\omega_z = 2\pi \times 11$ Hz. Typically, the size of a condensate is the order of few μm. Since $\sqrt{2}r/w < 1$, one can expand the Gaussian laser potential,

$$U(r) = U_0 e^{-\frac{2r^2}{w^2}} \sim U_0 \left(1 - \frac{2r^2}{w^2} + \frac{2r^4}{w^4}\right). \quad (3)$$

The resulting potential can be written as,

$$V_t(\tilde{r}) = \frac{\hbar\omega_0}{2}\left[(1-k)\tilde{r}^2 + \epsilon\left(\tilde{x}^2 - \tilde{y}^2\right) + \left(\frac{\omega_z^2}{\omega_0^2}\right)\tilde{z}^2 + \lambda\tilde{r}^4\right],$$

where the co-ordinates $\tilde{r}, \tilde{x}, \tilde{y}$, and $\tilde{z}$ are in units of the harmonic oscillator length $a_0 = \sqrt{\hbar/m\omega_0} = 1.475$ μm. Here, $k = 4U_0/m\omega_0^2 w^2$ and $\lambda = k(a_0/w)^2$. For the given values of the w, U_0 and $\omega_0 = 2\pi \times 75.5$ Hz, $k = 0.24$ and $\lambda = 8.297 \times 10^{-4}$ at the ENS experiment [15]. The value of k renormalize the oscillator frequency (ω_0) of the magnetic trap and also the anharmonic term λ depends on the value of k. At the ENS experiment, k is small. In this paper, we are interested to study when k is large. When $k < 1$, $k = 1$ and $k > 1$, the effective potential is quadratic-plus-quartic potential, only quartic potential and quartic-minus-quadratic potential, respectively. If $k > 1$, the effective external potential becomes Mexican hat structure. In principle, $k \geq 1$ can be easily obtain in the current experimental set up at ENS by increasing U_0 from $U_0 = 1.242 \times 10^{-30}$ J to $U_0 \geq 5 \times 10^{-30}$ J. For simplicity, we study quasi-2D system since the elongated condensate expands radially and contracts axially when it rotates rapidly. The equation of motion of the condensate wave function $\psi(\vec{r})$ is described by the mean-field Gross-Pitaevskii equation,

$$i\hbar\frac{\partial\psi(\vec{r})}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_t(\vec{r}) + g_2|\psi(\vec{r})|^2 - \Omega_0 L_z\right]\psi(\vec{r}).$$

Here, $g_2 = 2\sqrt{2\pi}\hbar\omega_z a a_z$ [16] is the strength of the mean-field interaction, a is the s-wave scattering length and $a_z = \sqrt{\frac{\hbar}{m\omega_z}}$. Also, $L_z = xp_y - yp_x$ is the z-component of the angular momentum operator and Ω_0 is the trap rotation frequency. When $\epsilon = 0$, it preserves the circular symmetry of the system.

One can write down the Lagrangian density corresponding to the quasi-2D system as follows:

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + V_t(x,y)|\psi|^2 + \frac{g_2}{2}|\psi|^4 - \Omega\psi^* L_z\psi\right). \quad (4)$$

Here, we use the time-dependent variational method to study the properties of a rotating BEC confined in a quadratic-plus-laser potential. In order to obtain the evolution of the condensate we assume the most general Gaussian wave function,

$$\psi\left(\tilde{x}, \tilde{y}, t\right) = C(t)e^{-\frac{1}{2}\left[\alpha(t)\tilde{x}^2 + \beta(t)\tilde{y}^2 - 2\gamma(t)\tilde{x}\tilde{y}\right]}, \quad (5)$$

where $C(t)$ is the normalization constant. Further, $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$, $\beta(t) = \beta_1(t) + i\beta_2(t)$ and $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ are the time-dependent dimensionless complex variational parameters. The α_1 and β_1 are inverse square of the condensate widths in x- and y-directions, respectively. The above mentioned order parameter describes only the vortex free condensate (irrotational system) as the phase ($S(x,y) = \gamma xy$) corresponds to the irrotational velocity flow.

We obtain the variational Lagrangian L by substituting equation (5) into equation (4) and integrating the Lagrangian density over the space co-ordinates,

$$\frac{L}{N\hbar\omega_0} = \frac{1}{4D}\Big[-(\beta_1\dot{\alpha_2} + \alpha_1\dot{\beta_2} - 2\gamma_1\dot{\gamma_2}) + (\alpha_1+\beta_1)D + (\alpha_2^2+\gamma_2^2)\beta_1 + (\beta_2^2+\gamma_2^2)\alpha_1 - 2(\alpha_2+\beta_2)\gamma_1\gamma_2 + (1-k)(\alpha_1+\beta_1) + \epsilon(\beta_1-\alpha_1) + \frac{\lambda}{2}\frac{(3\alpha_1^2 + 2\alpha_1\beta_1 + 3\beta_1^2 + 4\gamma_1^2)}{D} + PD^{3/2} + 2\Omega(\gamma_1(\beta_2-\alpha_2) + \gamma_2(\alpha_1-\beta_1))\Big], \quad (6)$$

where $D = \sqrt{\alpha_1\beta_1 - \gamma_1^2}$ and $P = 2\sqrt{2/\pi}(N-1)a/a_z$ is the dimensionless parameter that indicates the strength of the mean-field interaction and it can be positive or negative depending on the sign of the s-wave scattering length a [14].

The variational energy of the rotating condensate at equilibrium is given in terms of the inverse square width of the condensate along the x- and y-directions and the phase parameter $\delta\gamma_2$,

$$\frac{E}{N\hbar\omega_0} = \frac{1}{4}\Bigg[(\alpha_1+\beta_1) + (\gamma_2^2 + 1 - k)\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right) + \epsilon\left(\frac{1}{\alpha_1} - \frac{1}{\beta_1}\right) + \frac{\lambda}{2}\left(\frac{3}{\alpha_1^2} + \frac{2}{\alpha_1\beta_1} + \frac{3}{\beta_1^2}\right) + P\sqrt{\alpha_1\beta_1} + 2\Omega\gamma_2\left(\frac{1}{\beta_1} - \frac{1}{\alpha_1}\right)\Bigg], \quad (7)$$

where $\Omega = \Omega_0/\omega_0$. We have used the fact that $\alpha_2 = \beta_2 = \gamma_1 = 0$ to obtain the above equation.

One can get the equilibrium value of the variational parameters, $\alpha_{10} = X$, $\beta_{10} = Y$ and $\gamma_{20} = Z$ by minimizing the energy with respect to the variational parameters,

$$X^2\left(1 + \frac{P}{2}\sqrt{\frac{Y}{X}}\right) = Z^2 - 2\Omega Z + 1 - k + \epsilon + \lambda\left(\frac{3}{X} + \frac{1}{Y}\right), \quad (8)$$

$$Y^2\left(1 + \frac{P}{2}\sqrt{\frac{X}{Y}}\right) = Z^2 + 2\Omega Z + 1 - k - \epsilon + \lambda\left(\frac{1}{X} + \frac{3}{Y}\right), \quad (9)$$

and
$$Z = \Omega \frac{(Y-X)}{(Y+X)}. \quad (10)$$
Equations (8), (9) and (10) describes how the shape of the condensate changes due to the change of the amplitude of the laser beam as well as external rotation.

The average angular momentum per particle is given by,
$$\frac{\langle L_z \rangle}{N\hbar} = \frac{1}{2} Z \left(\frac{1}{Y} - \frac{1}{X} \right) = \frac{\Omega}{2} \frac{(1-\eta)^2}{\eta(X+Y)}, \quad (11)$$
where $\eta = Y/X$ is the ratio of the square of the widths along the x- and y-directions. Equation (11) explicitly shows how the angular momentum is transfered to the trapped BEC due to the shape deformation with the rotation frequency. The vortex would appear when $\langle L_z \rangle = N\hbar$. Therefore, one can estimate the critical angular frequency to create a single vortex from the relation: $\langle L_z \rangle = N\hbar$.

First we calculate the quadrupole mode frequency of BEC confined in a magnetic-plus-laser potential. We expand the Lagrangian in the following way: $\alpha = X + \delta\alpha_1$, $\beta = Y + \delta\beta_1$, and $\gamma = \delta\gamma_1$. We keep only the second order deviations from their equilibrium values. Then we calculate the Lagrangian quadratic in the deviations and using the Euler-Lagrangian equation of motion, we get the following two coupled equations of $\delta\alpha_1$ and $\delta\beta_1$:
$$\delta\ddot{\alpha}_1 = \left[\frac{P}{2} X^2 \sqrt{\frac{Y}{X}} - 4(1-k+\epsilon) - 2\lambda \left(\frac{9}{X} + \frac{2}{Y} \right) \right] \delta\alpha_1 - \left[\frac{P}{2} X^2 \sqrt{\frac{X}{Y}} + 2\lambda \frac{X}{Y^2} \right] \delta\beta_1, \quad (12)$$
$$\delta\ddot{\beta}_1 = - \left[\frac{P}{2} Y^2 \sqrt{\frac{Y}{X}} + 2\lambda \frac{Y}{X^2} \right] \delta\alpha_1 + \left[\frac{P}{2} Y^2 \sqrt{\frac{X}{Y}} - 4(1-k-\epsilon) - 2\lambda \left(\frac{2}{X} + \frac{9}{Y} \right) \right] \delta\beta_1. \quad (13)$$
For an isotropic trap, $\epsilon = 0$, and we set $\delta\alpha_1 = -\delta\beta_1$ to calculate the lowest energy quadrupole mode frequency which is given by,
$$\omega_q^2 = 4(1-k) + \frac{20\lambda}{R_0} - PR_0^2, \quad (14)$$
where R_0 is the equilibrium radius of the system, and this can be obtained from the real solution of the cubic equation: $(1 + P/2)R_0^3 - (1-k)R_0 - 4\lambda = 0$. The above quadrupole frequency ω_q is valid for all interaction (repulsive, attractive) strength. The quadrupole mode frequency ω_q vs. P for different values of k, i.e. for different configuration of the effective potential, is shown in Figure 1.

Figure 1 shows that when $k < 1$, the ω_q decreases as P increases and after a critical value of P, ω_q start increasing slowly. On the other hands, for $k \geq 1$, ω_q increases as P increases. The quadrupole mode frequency ω_q vs. k for different values of P is shown in Figure 2. For $k = 0 = P$, it reproduces the known result for the quadrupole mode frequency of a harmonically trapped noninteracting BEC. With the increasing of k, ω_q of an ideal Bose gases is decreasing faster than the interacting cases.

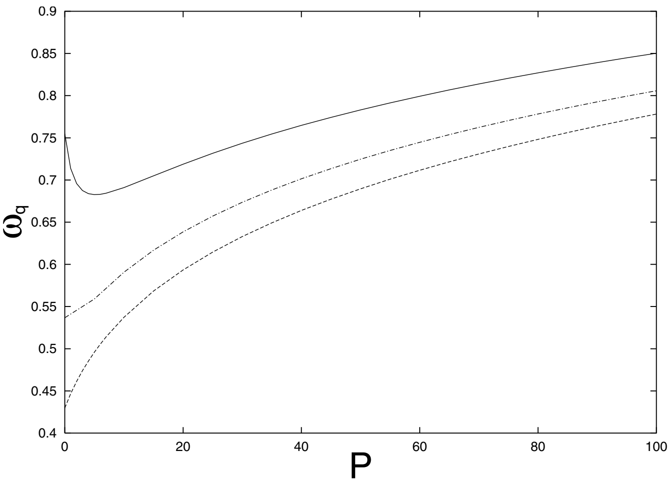

Fig. 1. Plots of quadrupole mode frequency ω_q vs. interaction parameter P for $k = 0.9, \lambda = 0.0031$ (upper curve), $k = 1.0, \lambda = 0.0034$ (middle curve) and $k = 1.1, \lambda = 0.0038$ (lower curve).

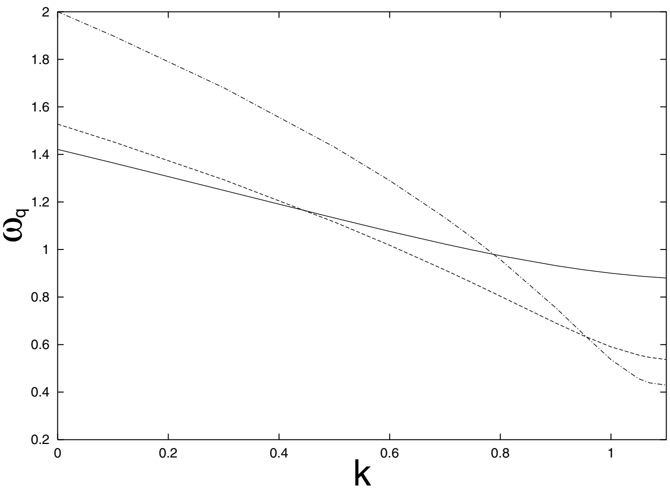

Fig. 2. Plots of quadrupole mode frequency ω_q vs. k for $P = 0$ (dot-dashed), 10 (dashed), 200 (solid).

The centrifugal term $-\Omega L_z$ shifts the quadrupole mode frequency by -2Ω. The quadrupole mode frequency with $m_z = -2$ is
$$\omega_{-2} = \sqrt{4(1-k) + \frac{20\lambda}{R_0} - PR_0^2} - 2\Omega. \quad (15)$$
We will discuss the effect of the harmonic-plus-Gaussian laser trap potential on the possibility of a vortex formation in a slowly rotating BEC. Using the three coupled equations (8), (9) and (10), we get the following two coupled

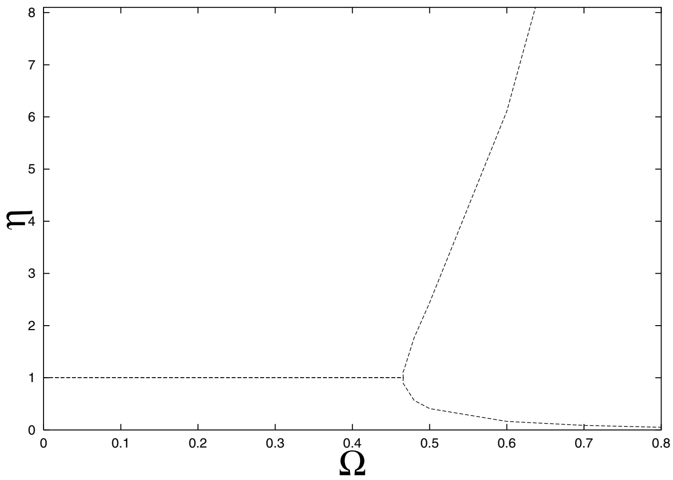

Fig. 3. The variation of η vs. the rotational frequency Ω for $P = 200$, $k = 0.9$, and $\epsilon = 0$.

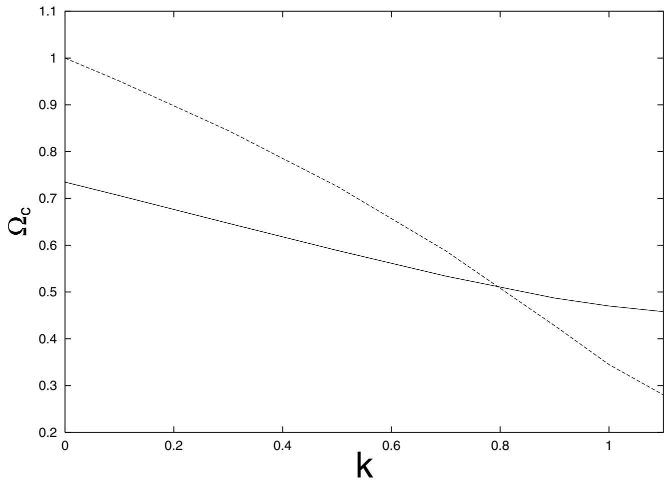

Fig. 4. Critical rotational frequency Ω_c vs. k for $P = 200$ (solid line) and $P = 0$ (dashed line).

polynomial equations:

$$0 = (X^3Y^3 + 2X^4Y^2 + X^5Y)\left(1 + \frac{P}{2}\sqrt{\frac{Y}{X}}\right) - X^3Y(3\Omega^2 + 1 - k + \epsilon) + (XY^3 + 2X^2Y^2) \times (\Omega^2 - 1 + k - \epsilon) - \lambda(5X^2Y + 7XY^2 + X^3 + 3Y^3) \quad (16)$$

and

$$0 = \left(X^3Y^3 + 2X^2Y^4 + XY^5\right)\left(1 + \frac{P}{2}\sqrt{\frac{X}{Y}}\right) - XY^3(3\Omega^2 + 1 - k + \epsilon) + (X^3Y + 2X^2Y^2) \times (\Omega^2 - 1 + k - \epsilon) - \lambda(5XY^2 + 7X^2Y + 3X^3 + Y^3). \quad (17)$$

For simplicity, we put $\epsilon = 0$ and solve those two coupled equations numerically. We find that spontaneous deformation occurs when the rotational frequency is greater than one-half of the quadrupole mode frequency ω_q, and start transferring the angular momentum to the system [see Eq. (11)]. This circular symmetry breaking is due to the tendency against an instability of the lowest energy surface mode frequency with $m_z = -2$. The variation of η with the rotational frequency Ω for $\epsilon = 0$ and $k = 0.9$ is shown in Figure 3. The same analysis can also be done for other values of k and P.

For $k = 0$ (only harmonic potential) and P is large, one can produce a vortex when $\Omega \sim 0.7$. For $k \neq 0$, the critical rotational frequency (Ω_c) to create a single vortex is small ($\Omega_c < 0.7$) compared to harmonic trap case ($\Omega_c \sim 0.7$). By adjusting the strength of the Gaussian laser beam, one can produce a vortex in a slowly rotating BEC. For a purely harmonic trapped ideal Bose gases, the lowest energy surface mode frequency is $2\omega_0$ and hence instability will occur when $\Omega = \omega_0$ at which the system is destabilized. Therefore, there is no spontaneous shape deformation and hence a vortex can not appear in the noninteracting gas confined by harmonic trap only. When $k \neq 0$, the the lowest energy surface mode frequency becomes less than $2\omega_0$ and an instability will occur when $\Omega < \omega_0$ at which the system is stabilized. There is a spontaneous shape deformation and one can produce a vortex in a slowly rotating noninteracting Bose gases confined in a harmonic-plus-Gaussian laser trap.

Figure 4 shows how Ω_c is decreasing with the parameter k for different values of P. As k increases, Ω_c is also decreasing. Hence there is a possibility of vortex formation of a slowly rotating BEC if k is finite. When $k < 0.8$, Ω_c is small in the Thomas-Fermi regime compared to the non-interacting case. When $k > 0.8$, Ω_c is large in the Thomas-Fermi regime compared to the non-interacting case. It implies that the two-body interaction helps to create a vortex when $k < 0.8$, but the two-body interaction do not help when $k > 0.8$.

In this work, we have shown that by adjusting amplitude of the Gaussian laser beam one can make quadratic-plus-quartic potential, purely quartic potential and quartic-minus-quadratic potential. We have also shown that an interacting BEC confined in a harmonic-plus-Gaussian laser tarp breaks the rotational symmetry of the Hamiltonian due to the instability of the lowest energy surface mode. It is argued that by increasing the amplitude of the Gaussian laser trap, a vortex can be created in a slowly rotating interacting as well as noninteracting BECs.

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